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Real hypersurfaces in the complex quadric with Reeb invariant Ricci tensor

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ABSTRACT

We introduce the notion of Reeb invariant Ricci tensor for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. The Reeb invariant Ricci tensor implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then according to each case, we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with Reeb invariant Ricci tensor.

1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [1–4] and [5]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ and they have rank 2.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give an example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Klein [6], and Smyth [7]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [8]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [6] and Reckziegel [9]).

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications related to commuting Ricci tensor were investigated by Kimura [10,11], Pérez [12] and Pérez and Suh [13,14] respectively. The classification problems of the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ with certain geometric

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conditions were mainly discussed in Suh [3,15] and [4], where the classification of *contact hypersurfaces*, *parallel Ricci tensor* and *harmonic curvature* of a real hypersurface in $G_2(\mathbb{C}^{m+2})$ were extensively studied. Moreover, in [4] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. Suh [5] strengthened this result to hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and gave a characterization of real hypersurfaces in $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ as follows:

Theorem A. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, Suh and Woo [16] have investigated a classification problem of real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with parallel Ricci tensor. Moreover, Suh [5] studied another classification for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ with commuting Ricci tensor as follows:

Theorem B. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

It is known that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$ in [5] and [17]. Moreover, in [4] we asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. Here, the Reeb flow on real hypersurfaces in $SU_{m+2}/S(U_mU_2)$ or $SU_{2,m}/S(U_2U_m)$ is said to be *isometric* if the shape operator commutes with the structure tensor. Then naturally it can be easily checked that the Ricci tensor commutes with the structure tensor. In the paper [18] due to Suh and Hwang, we investigated this problem for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and obtained the following result:

Theorem C. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with commuting Ricci tensor. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^\perp$, then M is locally congruent to an open part of a tube around totally geodesic $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$ or M has 3 distinct constant principal curvatures given by*

$$\alpha = \sqrt{2(m-3)},\ \gamma = 0,\ \lambda = 0,\ \textit{and}\ \mu = -\frac{2}{\sqrt{2(m-3)}}\ \textit{or}$$
$$\alpha = \sqrt{\frac{2}{3}(m-3)},\ \gamma = 0,\ \lambda = 0,\ \textit{and}\ \mu = -\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi],\ T_\gamma = [A\xi, AN],\ \phi(T_\lambda) = T_\mu,\ \textit{and}\ \dim\ T_\lambda = \dim\ T_\mu = m-2.$$

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}_z$ of T_zM, $z \in M$, defined by

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z \in M$. Thus for a case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Thus for a case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$. Moreover, the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then motivated by the above result, in [17] we gave another theorem for real hypersurfaces in the complex quadric Q^m with parallel Ricci tensor and $\mathfrak{A}$-isotropic unit normal.

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

When we consider a hypersurface M in the complex quadric Q^m, under the assumption of some geometric properties the unit normal vector field N of M in Q^m can be divided into two classes if either N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [19] and [17]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in [19] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [17]).

In the study of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ or complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2\cdot U_m)$ we studied hypersurfaces with parallel Ricci tensor and gave non-existence properties respectively (see [3] and [16]). In [17] we also considered the notion of parallel Ricci tensor, that is, $\nabla\mathrm{Ric}=0$, for hypersurfaces M in Q^m. But from the assumption of Ricci parallel, it was impossible to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [17] we gave a classification with the further assumption of $\mathfrak{A}$-isotropic.

But fortunately when we consider Ricci commuting, that is, $\mathrm{Ric}\cdot\phi=\phi\cdot\mathrm{Ric}$ for hypersurfaces M in Q^m, we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see Suh and Hwang [18]). Then motivated by such a result and using Theorem C, in this paper we give a complete classification for real hypersurfaces in the complex quadric Q^m with Reeb invariant Ricci tensor, that is, $\mathcal{L}_\xi Ric=0$ as follows:

Main Theorem. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m\geq 4$, with Reeb invariant Ricci tensor. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^\perp$, then M is locally congruent to an open part of a tube around totally geodesic $\mathbb{C}P^k$ in Q^{2k}, $m=2k$ or M has 3 distinct constant principal curvatures given by*

$$\alpha=\sqrt{2(m-3)},\ \gamma=0,\ \lambda=0,\ \text{and } \mu=-\frac{2}{\sqrt{2(m-3)}}\ \text{or}$$
$$\alpha=\sqrt{\frac{2}{3}(m-3)},\ \gamma=0,\ \lambda=0,\ \text{and } \mu=-\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces respectively

$$T_\alpha=[\xi],\ T_\gamma=[A\xi,AN],\ \phi(T_\lambda)=T_\mu,\ \text{and } \dim\,T_\lambda=\dim\,T_\mu=m-2.$$

Remark 1.1. In Main Theorem the second and third ones can be explained geometrically as follows: the real hypersurface M is locally congruent to $M_1\times\mathbb{C}$, where M_1 is a tube of radius $r=\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{m-3}$ or respectively, of radius $r=\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{m-3}{3}}$, over an $m-1$-dimensional unit sphere S^{m-1} in Q^{m-1}. Then, by the result due to Suh [17], M_1 becomes a contact hypersurface defined by $S\phi+\phi S=k\phi$, $k=-\frac{2}{\sqrt{2(m-3)}}$, and $k=-\frac{\sqrt{6}}{\sqrt{m-3}}$ respectively. By using the Segre embedding, the embedding $M_1\times\mathbb{C}\subset Q^{m-1}\times\mathbb{C}\subset Q^m$ is defined by $(z_0,z_1,\cdots,z_m,w)\to(z_0w,z_1w,\cdots,z_mw,0)$. Here $(z_0w)^2+(z_1w)^2+\cdots+(z_mw)^2=(z_0^2+\cdots+z_m^2)w^2=0$, where $\{z_0,\cdots,z_m\}$ denotes a coordinate system in Q^{m-1} satisfying $z_0^2+\cdots+z_m^2=0$.

2. The complex quadric

For more background to this section we refer to [6,8,9,17,19] and [20]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2+\cdots+z_{m+1}^2=0$, where $z_0,\ldots,z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini–Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini–Study metric $\bar{g}$ is defined by $\bar{g}(X,Y)=\Phi(JX,Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi=-4i\partial\bar{\partial}\log f_j$ on an open set $U_j=\{[z^0,z^1,\cdots,z^{m+1}]\in\mathbb{C}P^{m+1}|z^j\neq 0\}$, where the function f_j denotes $f_j=\sum_{k=0}^{m+1}t_j^k\bar{t}_j^k$, and $t_j^k=\frac{z^k}{z^j}$ for $j,k=0,\cdots,m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J,g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1}=SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o=[0,\ldots,0,1]\in\mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2}\subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1}\subset\mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m=SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m\geq 3$ from now on.

In another way, the complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi: S^{2m+3}\to\mathbb{C}P^{m+1},\quad z\to[z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for the complex quadric Q^m as follows:

$$\begin{array}{ccc}\tilde{Q}=\pi^{-1}(Q) & \xrightarrow{\ \tilde{i}\ } & S^{2m+3}\subset\mathbb{C}^{m+2}\\ \pi\downarrow & & \pi\downarrow\\ Q=Q^m & \xrightarrow{\ i\ } & \mathbb{C}P^{m+1}\end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is given by

$$\tilde{Q} = \left\{ x + iy \in \mathbb{C}^{m+2} | g(x, x) = g(y, y) = \frac{1}{2} \text{ and } g(x, y) = 0 \right\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2})$ and $y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + iy \in \tilde{Q}$ respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^{\perp}$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$, and F_z and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ becomes a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + iy \in \tilde{Q}$ it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} | \quad g(x, u) + g(y, v) = 0, \quad g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z | \quad g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_* H_z = T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)}Q$ respectively. This gives that at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see Reckziegel [9]).

Then let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}} w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]}Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}} w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda\bar{w} \\ &= \lambda A_{\bar{z}} \lambda\bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X, Y)Z = &\, g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &+ g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY. \end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. When $W = X$ for $X \in V(A)$, $t = 0$, there exist many kinds of maximal 2-flats $\mathbb{R}X + \mathbb{R}Z$ for $Z \in V(A)$ orthogonal to $X \in V(A)$. So the tangent vector X is said to be singular. When $W = (X + JY)/\sqrt{2}$ for $t = \frac{\pi}{4}$, it becomes also a singular tangent vector, which belongs to many kinds of maximal 2-flats given by $\mathbb{R}(X + JY) + \mathbb{R}Z$ for any $Z \in V(A)$ orthogonal to $X \in V(A)$ or $\mathbb{R}(X + JY) + \mathbb{R}JZ$ for any $JZ \in JV(A)$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

Lemma 3.1 ([19]). *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies the following

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurfaces M with the smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider the transform JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the equation of Codazzi

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ & + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.2}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [9]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 = & \, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.4}$$

4. Reeb invariance and a key lemma

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned} R(X, Y)Z = & \, g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY + g(SY, Z)SX - g(SX, Z)SY \end{aligned}$$

for any $X, Y, Z \in T_zM, z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_zQ^m, z \in M, \rho(X) = g(AX, N)$, where BX and $\rho(X)N$ respectively denote the tangential and the normal component of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi, N) = 0$. Then it follows that

$$\begin{aligned} AN &= AJ\xi = JA\xi = -J(B\xi + \rho(\xi)N) \\ &= -(\phi B\xi + \eta(B\xi)N). \end{aligned}$$

The equation gives $g(AN, N) = -\eta(B\xi)$ and $g(AN, \xi) = 0$. From this, together with the definition of the Ricci tensor, we have

$$Ric(X) = (2m-1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN + g(AX, \xi)A\xi + (TrS)SX - S^2X. \tag{4.1}$$

On the other hand, it is known that the Ricci tensor is Reeb invariant, that is, $\mathcal{L}_\xi S = 0$ if and only if

$$(\phi S - S\phi) \cdot Ric = Ric \cdot (\phi S - S\phi). \tag{4.2}$$

Here we want to give a remark as follows:

Remark 4.1. Let M be a real hypersurface over a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$, $m = 2k$. Then by a theorem due to Suh [17] the structure tensor commutes with the shape operator, that is, $S\phi = \phi S$. Moreover, the unit normal vector field N becomes $\mathfrak{A}$-isotropic. This gives $\eta(B\xi) = g(A\xi, \xi) = 0$. So it naturally satisfies the formula (4.2), that is, Ricci commuting.

On the other hand, from (4.3) we assert an important lemma as follows:

Lemma 4.2. *Let M be a Hopf real hypersurface in Q^m, $m \geq 3$, with Reeb invariant Ricci tensor. Then the unit normal vector field N becomes singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. By putting $X = \xi$ in (4.2) we get

$$(\phi S - S\phi)Ric(\xi) = 0. \tag{4.3}$$

Here from (4.1) the Ricci curvature is given by

$$Ric(\xi) = (2m-4)\xi - g(AN, N)A\xi + g(A\xi, \xi)A\xi + (trS)\alpha\xi - \alpha^2\xi,$$

where $g(A\xi, \xi) = g(AJN, JN) = -g(JAN, JN) = -g(AN, N)$. Substituting this one into (4.3) gives

$$g(AN, N)(\phi S - S\phi)A\xi = 0. \tag{4.4}$$

The first case gives that $g(AN, N) = g(A\xi, \xi) = \cos 2t = 0$, that is, $t = \frac{\pi}{4}$. This implies that the unit normal N becomes $N = \frac{X+JY}{\sqrt{2}}$, which means that N is $\mathfrak{A}$-isotropic.

The second case gives that

$$\phi SA\xi = S\phi A\xi. \tag{4.5}$$

Similarly, we also know that

$$\phi S(AN)^T = S\phi(AN)^T, \tag{4.6}$$

where $(AN)^T$ denotes the tangential component of the vector field AN in Q^m. From these two Eq. (4.5) and we know that the shape operator S commutes with the structure tensor ϕ on the distribution $Q^\perp = Span[A\xi, (AN)^T]$.

On the other hand, by taking the inner product of (4.5) with the tangent vector field $A\xi$ we know that

$$S\phi A\xi = \phi SA\xi = 0. \tag{4.7}$$

This gives that

$$SA\xi = \alpha\eta(A\xi)\xi. \tag{4.8}$$

By virtue of the commuting $S\phi = \phi S$ on the distribution $\mathcal{Q}^\perp = [A\xi, (AN)^T]$, we know that $\lambda = 0$ or $\lambda = \alpha$ if we put $SAN = \lambda AN$. Moreover, in a paper due to Suh [17] we have mentioned that the distribution $\mathcal{Q}^\perp$ is invariant under the shape operator S if and only if $\phi S = S\phi$ on the distribution $\mathcal{Q}^\perp$. Then, together with the notion of Hopf, without loss of generality we may put

$$S\xi = \alpha\xi, \quad SA\xi = \alpha A\xi, \quad SAN = \alpha AN.$$

From this, together with (4.8), we have for a non-vanishing Reeb function $\alpha \neq 0$

$$A\xi = \eta(A\xi)\xi = \pm\xi.$$

When the Reeb function α is vanishing, by the formula in Section 3, that is,

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi),$$

it follows that

$$g(Y, (AN)^T)g(\xi, A\xi) = 0.$$

Since in the second case we have assumed that N is not $\mathfrak{A}$-isotropic, we know $g(\xi, A\xi) \neq 0$. So it follows that $(AN)^T = 0$. This means that

$$AN = (AN)^T + g(AN, N)N = g(AN, N)N.$$

Then it implies that

$$N = A^2N = g(AN, N)AN = g^2(AN, N)N.$$

This gives that $g(AN, N) = \pm 1$, that is, we can take the unit normal N such that $AN = N$. So the unit normal N is $\mathfrak{A}$-principal, that is, $AN = N$. $\square$

In order to prove our main theorem in the introduction, by virtue of Lemma 4.2, we can divide into two classes of hypersurfaces in Q^m with the unit normal N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. When M is with $\mathfrak{A}$-isotropic, in Section 5 we will give its proof in detail and in Section 6 we will give the remainder proof for the case that M has a $\mathfrak{A}$-principal normal vector field.

5. Proof of main theorem with $\mathfrak{A}$-isotropic

In this section we want to prove our Main Theorem for real hypersurfaces M in Q^m with commuting Ricci tensor when the unit normal vector field becomes $\mathfrak{A}$-isotropic.

Since we assumed that the unit normal N is $\mathfrak{A}$-isotropic, by the definition in Section 3 we know that $t = \frac{\pi}{4}$. Then by the expression of the $\mathfrak{A}$-isotropic unit normal vector field, (3.3) gives $N = \frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}JZ_2$. This implies that $g(A\xi, \xi) = 0$. Since the unit normal N is $\mathfrak{A}$-isotropic, we know that $g(\xi, A\xi) = 0$. Moreover, by (3.4) and using an anti-commuting property $AJ = -JA$ between the complex conjugation A and the Kähler structure J, we proved the following (see also Lemma 4.2 in [19]).

Lemma 5.1. *Let M be a Hopf hypersurface in Q^m with (local) unit normal vector field N. For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi) \\ & + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X, Y on M.

Then for $\mathfrak{A}$-isotropic unit normal the Ricci tensor S of a real hypersurface M in the complex quadric Q^m becomes

$$Ric(X) = (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X.$$

From this, together with the fact that $A\xi = \phi AN$ and $\phi A\xi = -AN$, it follows that

$$\phi \cdot Ric(X) = (2m-1)\phi X + g(AX, N)A\xi - g(AX, \xi)AN + h\phi SX - \phi S^2X \tag{5.1}$$

and

$$Ric(\phi X) = (2m-1)\phi X - g(X, A\xi)AN + g(X, AN)A\xi + hS\phi X - S^2\phi X, \tag{5.2}$$

where the function h denotes the trace of the shape operator S of M in Q^m. Then subtracting (5.2) from (5.1) gives the following

$$\phi \cdot Ric(X) - Ric(\phi X) = h(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X. \tag{5.3}$$

On the other hand, we know that the Reeb invariant Ricci tensor $\mathcal{L}_\xi Ric = 0$ is equivalent to the following

$$(\phi S - S\phi) \cdot Ric = Ric \cdot (\phi S - S\phi). \tag{5.4}$$

By using the formula (5.4) and taking the trace to (5.3), we have

$$\begin{aligned} Tr(\phi \cdot Ric - Ric \cdot \phi)^2 &= \sum_{i,j} g(\phi \cdot Ric(e_i) - Ric \cdot \phi(e_i), \phi \cdot Ric(e_i) - Ric \cdot \phi(e_i)) \\ &= hTr(\phi S - S\phi)(\phi \cdot Ric - Ric \cdot \phi) + Tr(\phi S^2 - S^2\phi)(\phi \cdot Ric - Ric \cdot \phi) \\ &= -Tr(\phi S^2 - S^2\phi)(\phi Ric - Ric\phi), \end{aligned} \tag{5.5}$$

where in the second equality we have used (5.4) and

$$\begin{aligned} Tr(\phi S - S\phi)(\phi \cdot Ric - Ric \cdot \phi) &= Tr\phi \cdot Ric(\phi S - S\phi) - Tr(\phi S - S\phi)Ric \cdot \phi \\ &= Tr\phi(\phi S - S\phi) \cdot Ric - Tr(\phi S - S\phi)Ric \cdot \phi \\ &= Tr(\phi S - S\phi)Ric \cdot \phi - Tr(\phi S - S\phi)Ric \cdot \phi \\ &= 0. \end{aligned}$$

On the other hand, the final term in (5.5) becomes the following

$$\begin{aligned} Tr(\phi S^2 - S^2\phi)(\phi \cdot Ric - Ric \cdot \phi) &= Tr\phi S^2\phi \cdot Ric - TrS^2\phi^2 \cdot Ric - Tr\phi S^2 Ric \cdot \phi + TrS^2\phi \cdot Ric \cdot \phi \\ &= 2Tr\phi S^2\phi \cdot Ric - TrS^2\phi^2 \cdot Ric - Tr\phi S^2 Ric \cdot \phi. \end{aligned} \tag{5.6}$$

By the property (5.4) due to the Reeb invariant Ricci tensor $\mathcal{L}_\xi Ric = 0$, we have

$$\phi S(\phi S \cdot Ric - Ric \cdot \phi S + Ric \cdot S\phi - S\phi Ric) = 0.$$

From this, by taking the trace, the first two terms become

$$Tr(\phi S)^2 \cdot Ric - Tr\phi S \cdot Ric \cdot \phi S = Tr(\phi S)^2 Ric - Tr(\phi S)^2 Ric = 0.$$

Then taking the trace of the other two terms becomes

$$Tr\phi S \cdot Ric \cdot S\phi = Tr\phi S^2\phi \cdot Ric.$$

By virtue of this equation and using the notion of Hopf, the formula (5.5) can be changed as follows:

$$\begin{aligned} Tr(\phi \cdot Ric - Ric \cdot \phi)^2 &= -Tr(\phi S^2 - S^2\phi)(\phi \cdot Ric - Ric \cdot \phi) \\ &= Tr\phi^2 \cdot Ric \cdot S^2 + Tr\phi^2 S^2 \cdot Ric - 2Tr\phi^2 S \cdot Ric \cdot S \\ &= 0, \end{aligned} \tag{5.7}$$

where we have used the following equations

$$\begin{aligned} Tr\phi^2 \cdot Ric \cdot S^2 &= Tr(-Ric \cdot S^2 + \eta(Ric \cdot S^2)\xi) \\ &= -TrRic \cdot S^2 + \eta(Ric(S^2\xi)), \end{aligned} \tag{5.8}$$

$$\begin{aligned} Tr\phi^2 \cdot S^2 \cdot Ric &= Tr(-S^2 \cdot Ric + \eta(S^2 \cdot Ric)\xi) \\ &= -TrRic \cdot S^2 + \eta(S^2 \cdot Ric\xi), \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} -2Tr\phi^2 S \cdot Ric \cdot S &= -2Tr(-S \cdot Ric \cdot S + \eta(S^2 \cdot Ric)\xi) \\ &= 2TrS \cdot Ric \cdot S - 2\eta(S \cdot Ric(S\xi)). \end{aligned} \tag{5.10}$$

From this we conclude that the Ricci tensor Ric commutes with the structure tensor ϕ for a case where the unit normal N is $\mathfrak{A}$-isotropic. Then by a theorem due to Suh and Hwang [18], we give a complete classification in our main Theorem in the introduction.

6. Proof of main theorem with $\mathfrak{A}$-principal

In this section we want to prove our Main Theorem for real hypersurfaces in the complex quadric Q^m with commuting Ricci tensor and $\mathfrak{A}$-principal unit normal vector field. By the Ricci tensor given in the formula (4.1) for $\mathfrak{A}$-principal unit normal, we give the following

$$Ric(\phi X) = (2m-1)\phi X - g(AN, N)A\phi X + g(A\phi X, N)AN + hS\phi X - S^2\phi X, \tag{6.1}$$

and

$$\phi Ric(X) = (2m-1)\phi X - g(AN, N)\phi AX + g(AX, N)\phi AN + h\phi SX - \phi S^2X, \tag{6.2}$$

where the function h denotes the trace of the shape operator S of M in Q^m.

When we consider the unit normal N is $\mathfrak{A}$-principal, the unit normal N is invariant under the complex conjugation A in $\mathfrak{A}$, that is, $AN = N$ and $A\xi = -\xi$. By using such properties into (6.1) and (6.2), we have

$$\phi\cdot Ric(X) - Ric\cdot\phi(X) = -\phi AX + A\phi X + h(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X.$$

From this, together with $\mathcal{L}_\xi Ric = 0$, which is equivalent to $(\phi S - S\phi)\cdot Ric = Ric\cdot(\phi S - S\phi)$, we have

$$\begin{aligned} Tr(\phi\cdot Ric - Ric\cdot\phi)^2 &= hTr(\phi S - S\phi)(\phi\cdot Ric - Ric\cdot\phi)\\ &\quad - Tr(\phi S^2 - S^2\phi)(\phi\cdot Ric - Ric\cdot\phi) - Tr(\phi A - A\phi)(\phi\cdot Ric - Ric\cdot\phi). \end{aligned}$$

On the other hand, the complex conjugation is involutive and anti-commuting such that $AJ = -JA$, and the unit normal N is $\mathfrak{A}$-invariant, it follows that

$$\phi A = -A\phi.$$

From this, together with $A\xi = -\xi$, we have

$$\begin{aligned} Tr\phi A(\phi\cdot Ric - Ric\cdot\phi) &= -TrA\phi^2\cdot Ric - TrRic\cdot\phi^2 A\\ &= 2TrRic\cdot A - \eta(Ric(A\xi)) - \eta(A\cdot Ric(\xi))\\ &= 2\{TrRic\cdot A + \eta(Ric(\xi))\}. \end{aligned}$$

Then it follows that

$$\begin{aligned} Tr(\phi\cdot Ric - Ric\cdot\phi)^2 &= -Tr(\phi S^2 - S^2\phi)(\phi\cdot Ric - Ric\cdot\phi) - Tr(\phi A - A\phi)(\phi\cdot Ric - Ric\cdot\phi)\\ &= 2\eta(Ric\cdot S^2(\xi)) - 2\eta(S\cdot Ric\cdot S(\xi)) - 4Tr(Ric\cdot A) - 4\eta(Ric(\xi)). \end{aligned} \tag{6.3}$$

The Ricci tensor given in the formula (4.1) for $\mathfrak{A}$-principal unit normal, that is, $A\xi = -\xi$ gives the following

$$Ric(X) = (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2X,$$

and

$$Ric(\xi) = \{2(m-1) + h\alpha - \alpha^2\}\xi.$$

Then it follows that

$$Ric(e_i) = (2m-1)e_i - 2\eta(e_i)\xi - Ae_i + hSe_i - S^2e_i,$$

and

$$Ric(Ae_i) = (2m-1)e_i + 2\eta(e_i)\xi - e_i + hSAe_i - S^2Ae_i,$$

where we have taken an orthonormal basis $\{\xi, e_1, \cdots, e_{m-1}, \phi e_1, \cdots, \phi e_{m-1}\}$ of T_zM, $z \in M$, in Q^m such that $Ae_i = e_i$, $A\phi e_i = -\phi e_i$, $A\xi = -\xi$ and $AN = N$. So it follows that

$$\begin{aligned} Tr(Ric\cdot A) &= g(A\xi, Ric(\xi)) + \sum_{i=1}^{2m-2} g(Ae_i, Ric(e_i))\\ &= -g(\xi, Ric(\xi)) + \sum_{i=1}^{m-1} g(Ae_i, Ric(e_i)) + \sum_{i=1}^{m-1} g(A\phi e_i, Ric(\phi e_i)). \end{aligned}$$

Substituting these ones into (6.3) and using the orthonormal basis, we have

$$\begin{aligned} Tr(\phi\cdot Ric - Ric\cdot\phi)^2 &= -4\textstyle\sum_{i=1}^{m-1}\{g(Ric(e_i), e_i) - g(\phi e_i, Ric(\phi e_i))\}\\ &= -4\{Tr^*Ric + Tr^*\phi\cdot Ric\cdot\phi\}\\ &= -4\{Tr^*Ric + Tr^*\phi^2\cdot Ric\}\\ &= -4\{Tr^*Ric - Tr^*Ric\}\\ &= 0, \end{aligned} \tag{6.4}$$

where Tr^*Ric denotes $Tr^*Ric = \sum_{i=1}^{m-1} g(Ric(e_i), e_i)$ for the orthonormal basis $\{\xi, e_1, \cdots, e_{m-1}, \phi e_1, \cdots, \phi e_{m-1}\}$ of T_zM, $z \in M$, in Q^m. This concludes that even for the $\mathfrak{A}$-principal normal the Ricci tensor Ric commutes with the structure tensor ϕ, that is, $Ric \cdot \phi = \phi \cdot Ric$. Then by Theorem C due to Suh and Hwang [18], we give a complete classification of our main result.

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